Prop M_3 represented by

$$\operatorname{Spec} \mathbb{Z}[\tfrac{1}{3}, \zeta_3][\mu, \tfrac{1}{\mu^3-1}]$$

with universal object

$$\mathcal{E}: \quad X^3 + Y^3 + Z^3 = 3\mu XYZ \quad \subset \mathbb{P}^2$$

$$e = (-1, 1, 0)$$

$$s = (1, 0, -1) = \alpha(1,0)$$

$$t = (-1, \zeta_3, 0) = \alpha(0,1)$$

determines h_i's uniquely

Idea Given $(E, \alpha)/S$ take suitably <u>normalized</u>

$$h_0, h_s, h_{2s} \in \Gamma(\mathcal{O}_E(D)) \quad D = [e] + [t] + [2t]$$

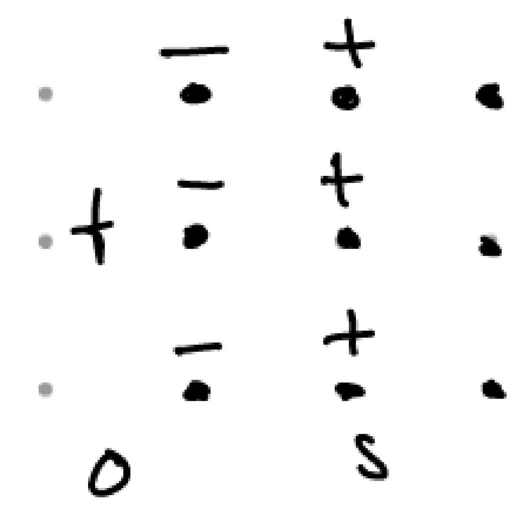
$\deg D = 3$, so $[h_0, h_s, h_{2s}]: E \hookrightarrow \mathbb{P}^2_S$

Then verify that image is $V(X^3 + Y^3 + \dots)$ for unique μ s.t. $\mu^3 - 1 \in \mathcal{O}_S^\times$.

Lem $(E,\alpha)/S$ EC + level-n-str, $n \geq 2$ $(\Rightarrow n \in \mathcal{O}_S^\times)$

$s = \alpha(1,0)$ $\quad t = \alpha(0,1)$

$$D_s := \sum_{j=0}^{n-1} \left([s+jt] - [jt] \right)$$



a) $\exists!\ h_s \in \pi_* \mathcal{O}_E(D_s)$ s.th. $D_s = \mathrm{div}(h_s)$

& $h_s(2s) = -1$

$\pi : E \to S$

b) $T_t^*(h_s) = e_n(s,t)\, h_s$.

Proof a) $\sum_{j=0}^{n-1} (s+jt) - jt = 0$

Abel's Thm $\big(=$ group homom. property of $E \to \hat{E}\big)$

$\Rightarrow \pi_* \mathcal{O}_E(D_s) \cong \mathcal{O}_E$ locally on S,

more precisely: $\mathcal{O}_E(D_s) = \pi^* \underbrace{\pi_* \mathcal{O}_E(D_s)}_{\mathcal{M}}$, $\mathcal{M} \in \mathrm{Pic}(S)$.

Since $\exists$ section $2s : S \to E$ not meeting D_s,

$\left(\Rightarrow \mathcal{O}_E(D_s)|_{[2s]} \cong (2s)_* \mathcal{O}_S \right)$

$\mathcal{M} \cong \mathcal{O}_S$, hence $\mathcal{O}_E(D_s) \cong \mathcal{O}_E$.

Existence of h_s now clear.

b) μ_n étale, so wlog $S = \operatorname{Spec} k$. $k = \bar{k}$

<u>Recall</u> Write $[n]^*([s]-[e]) = \operatorname{div}(g_s)$.

Then $e_n(s,t) = \dfrac{T_t^*(g_s)}{g_s}$

So our claim is $\dfrac{T_t^*(g_s)}{g_s} = \dfrac{T_t^*(h_s)}{h_s}$

$\iff$ g_s/h_s T_t-invariant.

Consider $E \xrightarrow{\varphi} E' := E/\langle t\rangle$.

To show: $g_s/h_s = \varphi^*(f)$ for some $f \in k(E')$.

Equivalent: $\exists\, D'$ principal divisor on E' s.th. $\varphi^*(D') = \operatorname{div}(g_s/h_s)$.

Let $n \cdot v = s$.

$$\operatorname{div}\left(\frac{g_s}{h_s}\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left([v+is+jt]-[is+jt]\right) - \sum_{j=0}^{n-1}\left([s+jt]-[jt]\right)$$

$$= -2\sum_j [s+jt] - \sum_{i=2}^{n-1}\sum_j [is+jt] + \sum_{i=0}^{n-1}\sum_j [v+is+jt]$$

$$= \varphi^*\Big(\underbrace{-2[\varphi(s)] - \sum_{i=2}^{n-1}[\varphi(is)] + \sum_{i=0}^{n-1}[\varphi(v+is)]}_{\circledast}\Big)$$

Now $\quad -2\varphi(s) - \sum_{i=2}^{n-1} i\,\varphi(s) + \sum_{i=0}^{n-1} \varphi(v + is)$

$= 0 \qquad$ (use $\; n\cdot\varphi(v) = \varphi(nv) = \varphi(s)$)

Abel's Thm $\Longrightarrow$ ⊛ is principal divisor. $\square$ Lem.

Inductively: $h_0 = 1$, $\quad h_{is} = h_s \cdot T_{-s}^*(h_{(i-1)s})$

Then $\quad \mathrm{div}(h_{is}) = \sum_{j=0}^{n-1} [is + jt] - [jt]$

(meaning single poles at $e, t, 2t, \ldots$)

hence $h_0, h_s, \ldots, h_{(n-1)s} \in \Gamma(E, \mathcal{O}(D))$

$D = [e] + [t] + [2t] + \cdots + [(n-1)t]$.

$T_t^* \circlearrowright \Gamma(E, \mathcal{O}(D))$ and h_{is} is eigenvector for $e_n(s,t)^i$.

<u>Note</u> $\xi = e_n(s,t)$ is primitive n-th root of 1, so $\Gamma(E, \mathcal{O}(D))$ decomposes uniquely into 1-dim eigenspaces for T_t^*-operation.

From now on $n=3$

$\varphi : (h_{2s}, h_s, h_0) : E \hookrightarrow \mathbb{P}^2$

We compute coordinates of $E[3] = \{ \varphi([is + jt]), \; i,j \in \mathbb{Z}/3 \}$

·) e : $h_0(e) = 1, \quad h_s(e) = \infty$

$$\frac{h_{2s}(e)}{h_s(e)} = (T_{-s}^* h_s)(e) = h_s(2s) = -1$$

So $\varphi(e) = (-1, 1, 0)$

·) s : $h_0(s) = 1, \quad h_s(s) = 0$

$$h_{2s}(s) = h_s(s) \cdot (T_{-s}^* h_s)(s) = h_s(s) \cdot h_s(e)$$

$$\text{"} = 0 \cdot \infty \text{"}$$

Cannot determine this scalar yet.

·) $2s$: $h_0(2s) = 1, \quad h_s(2s) = -1$

$$h_{2s}(2s) = h_s(2s) \cdot (T_{-s}^* h_s)(2s) = (-1) \cdot 0 = 0$$

$\varphi(2s) = (0, -1, 1)$

·) $e, s, 2s$ lie on a line because $e + s + 2s = e$.

Take form $aX + bY + cZ = 0$

$e = (-1, 1, 0)$, $2s = (0, -1, 1)$

$\Rightarrow a = b = c$

$\Rightarrow \psi(s) = (1, 0, -1)$

·) t : $h_s(t) = \infty$

$$\frac{h_{2s}(t)}{h_s(t)} = T_{-t}^{*}\left(\frac{h_{2s}}{h_s}\right)(e) = \frac{e_n(2s, -t)}{e_n(s, -t)} \cdot \frac{h_{2s}}{h_s}(e)$$

$$= e_n(s,t)^{-1} \cdot (-1)$$

Put $\zeta = e_n(s,t)$. Hence

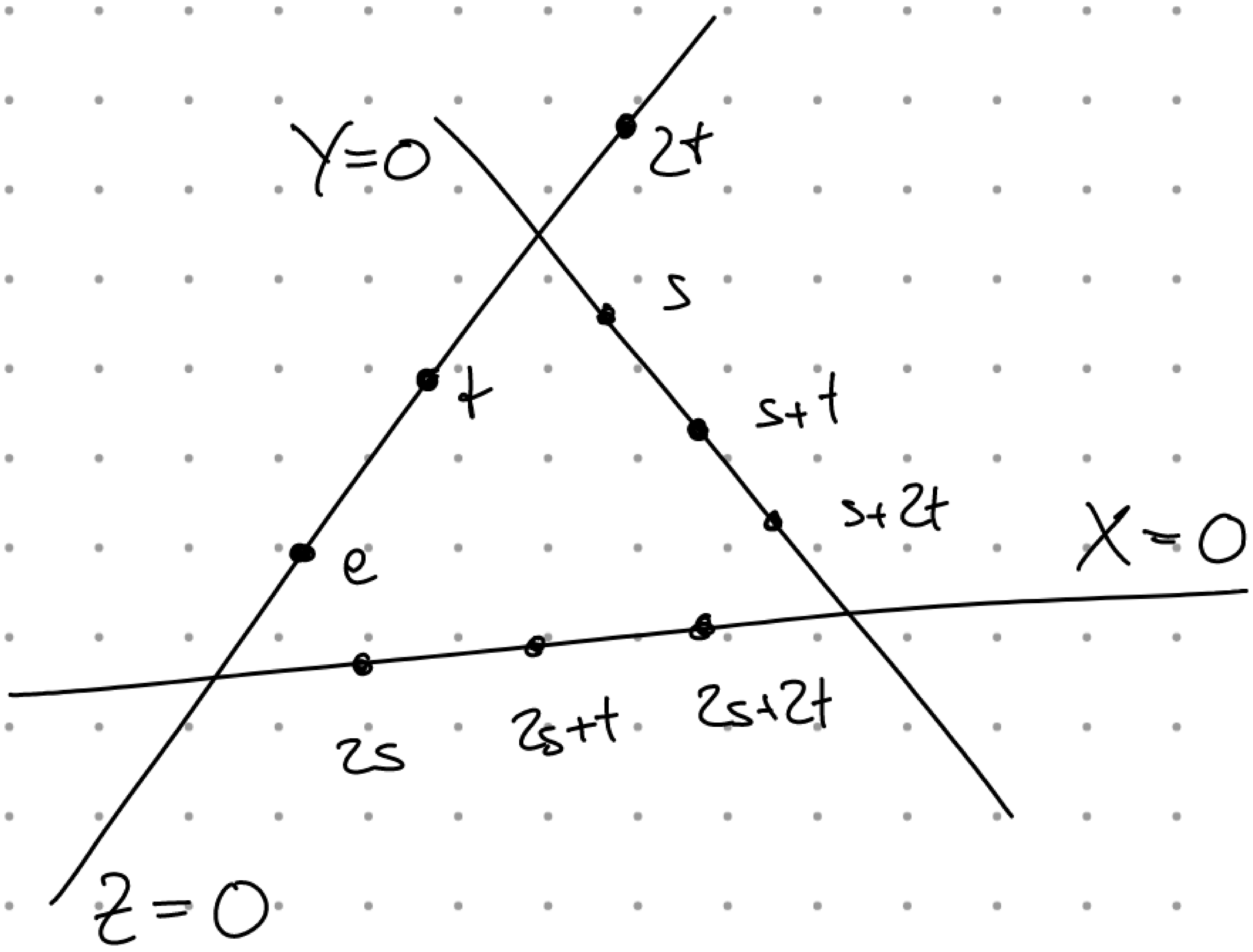
$\psi(t) = (-\zeta^{-1}, 1, 0) = (-1, \zeta, 0)$

Same ideas provide

$2t = (\zeta, -1, 0)$

$s + t = (-1, 0, \zeta)$, $s + 2t = (\zeta, 0, -1)$

$2s + t = (0, \zeta, -1)$, $2s + 2t = (0, -1, \zeta)$



Now $\psi(E) \subset \mathbb{P}^2$ described by a cubic equation

$$F = \sum a_{ijk} X^i Y^j Z^k$$

Put $Z=0$. Obtain

$$F(X,Y,0) = a_{300} X^3 + a_{030} Y^3 + a_{21} X^2 Y + a_{12} X Y^2$$

Now $e = (1,-1,0)$, $t = (-1, \zeta, 0)$, $2t = (\zeta, -1, 0)$

are precisely zeros of $X^3 + Y^3$ $\Rightarrow$ $a_{21} = a_{12} = 0$

$$a_{300} = a_{030}$$

Same w/ $X=0$, $Y=0$

$$\Rightarrow F = a(X^3 + Y^3 + Z^3) + b\, XYZ.$$

Since non-singular, $a \neq 0$

so wlog, $F = X^3 + Y^3 + Z^3 - 3\mu XYZ$.

Claim $\mu^3 - 1 \in \mathcal{O}_S^\times$

wlog $S = \operatorname{Spec} k$. Consider chart $Z \neq 0$.

Jacobi matrix:

$$\frac{\partial}{\partial x} = 3x^2 - 3\mu y \qquad x = \frac{X}{Z}, \quad y = \frac{Y}{Z}.$$

$$\frac{\partial}{\partial y} = 3y^2 - 3\mu x$$

If $\mu^3 = 1$, then $x = y = \mu$ solves

$$F(x, y, 1) = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \implies \text{singular}.$$

So $\mu^3 \neq 1$.

Conversely If singular, may solve

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = F(x, y, 1) = 0$$

Obtain $\quad x^3 + y^3 + 1 - 3y^3 = 0$

$\qquad\qquad x^3 + y^3 + 1 - 3x^3 = 0$,

so $x^3 = y^3$. Also $x^3 = y^3 = 3\mu xy$.

So $x^3 = y^3 = 1$.

$\Rightarrow \mu^3 = \left(\frac{x^2}{y}\right)^3 = 1$. $\square$ Claim.

Note F symmetric, so computation also applies to charts $X \neq 0$, $Y \neq 0$.

Thus we see

1) $X^3 + Y^3 + Z^3 = 3\mu XYZ$

$e = (1,-1,0)$ is EC$/ \mathbb{Z}[\frac{1}{3}, \zeta_3][\mu, \frac{1}{\mu^3-1}] =: R$

$s = (1,0,-1)$ + two sections s,t.

$t = (-1, \zeta_3, 0)$

2) Given EC $(E,\alpha)/S$, α level-3-str,

$\exists! \; S \xrightarrow{u} R \quad (E, \alpha_1, \alpha_2) = u^*(\mathcal{E}, s, t)$

Exercise s,t form a level-3-str.

(R is int. dom., enough to check over Frac R.

Now use the following characterization of

3-torsion points as <u>flex points</u>

$3P = 0 \iff -P = 2P$

$\iff \exists$ line $L \subset \mathbb{P}^2$ s.th. $L \cap E$ only in P

(with multiplicity 3))

<u>Conclusion</u> $(\operatorname{Spec} R, \mathcal{E}, s, t)$ represent $\mathcal{M}_3$. $\blacksquare$

<u>Prop</u> $\mathcal{M}_4$ is represented by

$$\operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{2}, i, \sigma, \frac{1}{\sigma(\sigma^4-1)}\right]$$

w/ univ. object

$$\mathcal{E}: \quad Y^2 Z = X(X-Z)\left(X - \tfrac{1}{4}\left(\sigma + \tfrac{1}{\sigma}\right)^2 Z\right)$$

$$\alpha(1,0) = \left(\tfrac{1}{2}\left(\sigma + \tfrac{1}{\sigma}\right),\ i \cdot \frac{(\sigma^2+1)(\sigma-1)^2}{4\sigma^2}\right)$$

$$\alpha(0,1) = \left(\frac{(\sigma+i)^2}{2i\sigma},\ -\frac{(\sigma^2-1)(\sigma+i)^2}{4\sigma^2}\right)$$

<u>Rmk</u> In particular, the (coarse) moduli spaces for

level	1,	2,	3,	4
	j-line	Legendre	M_3	M_4

are all localizations of $\mathbb{A}^1$ after addition of ζ_3 resp. i.

§ Inflection points

<u>Abel's Thm</u> $E \to \hat{E}$, $x \mapsto \mathcal{O}([x]-[0])$ is iso.

<u>Reformulation</u> $\sum_i n_i x_i = 0$, $x_i \in E(S)$

$$\Leftrightarrow \bigotimes_i \mathcal{O}([x_i]-[0])^{n_i}$$

$$= \mathcal{O}\left(\sum n_i [x_i] - \left(\sum n_i\right)[0]\right) = \underbrace{0 \text{ in } \hat{E}}$$

i.e. LHS $\in \pi^* \mathrm{Pic}(S)$.

Special case: $\deg\left(\sum n_i x_i\right) = \sum n_i = 0$.

Then $\sum n_i x_i = 0 \Leftrightarrow \mathcal{O}\left(\sum n_i \cdot [x_i]\right) = 0$ in $\hat{E}$.

Condition on RHS is indep of choice of e!

(<u>Note</u> Rigidity $\Rightarrow$ Any map of curves $E \to E'$ underlying ECs E, E' is translation $\circ$ group homo.

$\Rightarrow$ LHS also a priori known to be indep of EC structure.)

Consequence $E \hookrightarrow \mathbb{P}^2_S$ any embedding

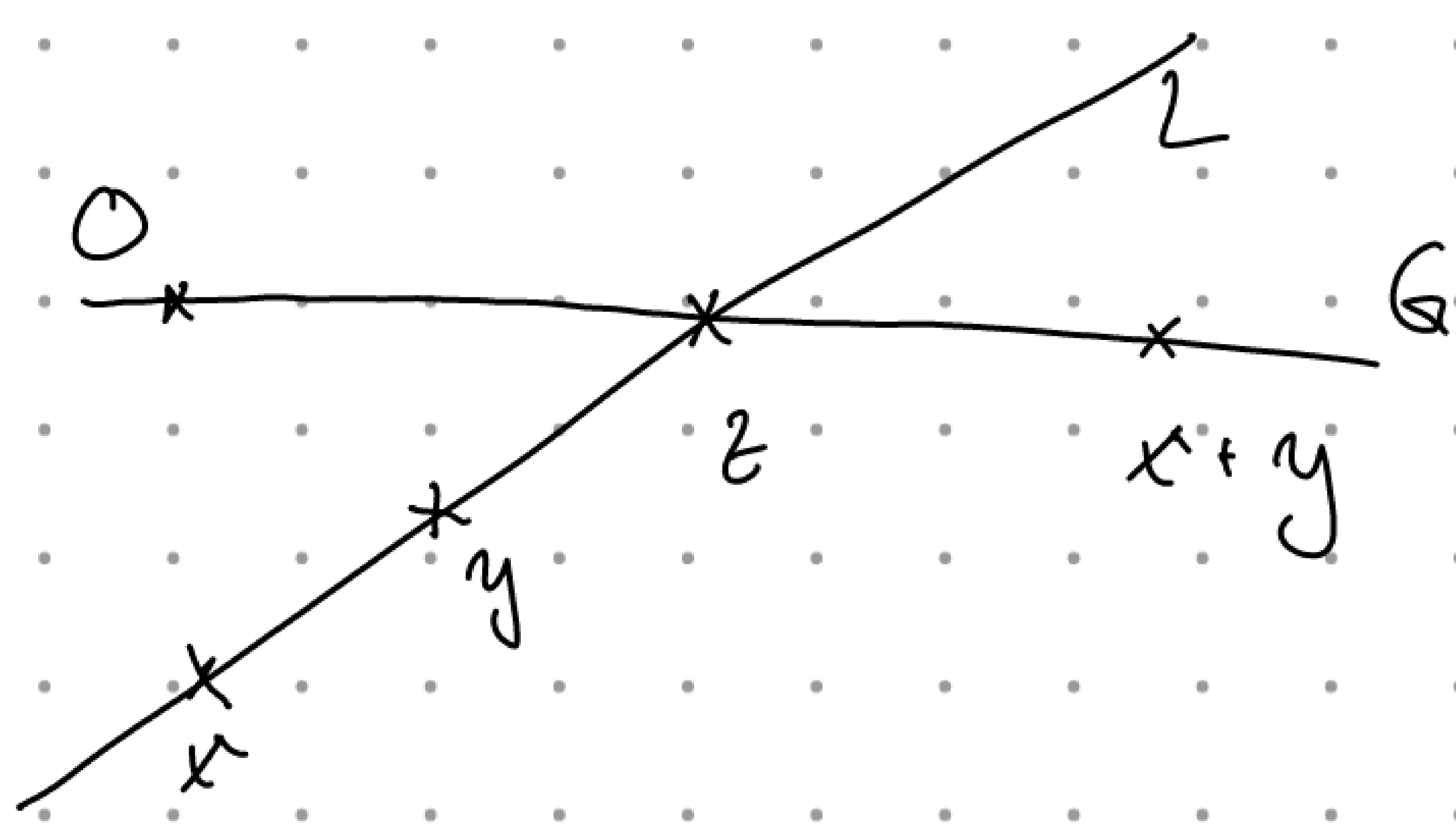
$H_1, H_2 \subset \mathbb{P}^2_S$ lines. Then $\mathcal{O}_{\mathbb{P}^2_S}(H_1 - H_2) \cong \mathcal{O}_{\mathbb{P}^2_S}$

Thus $\mathcal{O}_E(H_1|_E - H_2|_E) \cong \mathcal{O}_E$

If $\left.\begin{array}{l} H_1 \cap E = \{A, B, C\} \\ \& \; H_2 \cap E = \{R, S, T\} \end{array}\right\} \subset E(S)$,

then $A+B+C = R+S+T$ wrt. any EC str on E.

So geometrically, group law described as follows:



Given $O \in E(S)$ defining an EC-str & $x, y \in E(S)$.

$z :=$ 3rd intersection point of L through x, y w/ E.

$x+y$ 3rd int. point of G through O, z.

Namely $x+y+z = O+z+(x+y)$ according to above explanation.

Gretchenfrage When is $z = -x-y$?

Equivalent When is $z + (x + y) = 0$?

By given addition law we find:

·) line through z, $x+y$ intersects E in O

·) line through O, O (= tangent to E in O)

intersects E in $z + (x + y)$.

So $z + (x + y) = 0 \iff$ tangent in O intersects E w/ mult. 3.

i.e. O inflection point.

Assume O inflection point. Then

$3x = 0 \iff x$ inflection point.

Claim For our universal EC

$X^3 + Y^3 + Z^3 = 3\mu XYZ$, $e = (-1, 1, 0)$,

e is an inflection point.

Proof $e \in D_+(Y)$, $x = \frac{X}{Y}$, $z = \frac{Z}{Y}$.

$f(x, z) = x^3 + 1 + z^3 - 3\mu x z$

Jacobi matrix: $(3x^2 - 3\mu z, 3z^2 - 3\mu x)$

in $e = (-1, 0)$: $(3, 3\mu)$

Tangent in $(-1,0)$ $\quad x + \mu z = -1$

Plug $x = -(1+\mu z)$ into $f(x,z)$:

$$-(1+\mu z)^3 + z^3 + 1 + 3\mu z(1+\mu z)$$

$$= -\mu^3 z^3 - 3\mu^2 z^2 - 3\mu z - 1 + 1 + z^3 + 3\mu z + 3\mu^2 z^2$$

$= (1-\mu^3) z^3$ has triple zero at $z = 0$. $\square$

<u>Upshot</u> Can check s, t inflection points to check

$$3s = 3t = 0.$$

(The scheme $M_3 = \operatorname{Spec} \mathbb{Z}[\frac{1}{3}, \zeta_3][\mu, \frac{1}{\mu^3 - 1}]$ is an integral domain, so computation may be done generically, i.e. understanding flex points/fields suffices.)